POWER LAWS OF REDUCED TURBULENT FRICTION FOR POLYMER SOLUTIONS

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Power laws of the resistance of solutions to flow in pipes and in a boundary layer are discussed as approximations of logarithmic functions. Although the resulting description is less accurate, this is compensated by its simplicity.

The logarithmic distribution of the average velocity and the corresponding "logarithmic" laws of resistance play a fundamental role in the description of flow in pipes and in a boundary layer, both for a viscous liquid [1] and for polymer solutions [2]. Power functions, which are simpler, are also frequently used in both cases. Unlike the case of a viscous liquid, the coefficients of the power laws of resistance for polymer solutions turn out to depend both on the properties of the solution (viscosity, polymer concentration, etc.) and on the dimensions of the region of flow.

Turbulent Flow in Pipes. As has been found empirically [3], and as also follows from general considerations of dimensionality and similarity analysis [2], the distribution of the average velocity of flow of polymer solutions far from the walls and the axis of a large pipe is well described by a logarithmic formula having form (when the degradation of the solution is negligibly small)

$$\langle u^+ \rangle = 2.5 \ln \xi + 5.5, \quad \xi = z^+ (u_*/u_{*\mathrm{cr}})^\beta, \tag{1}$$

where $\beta > 0$ for $u_* > u_{*\mathrm{cr}}$ and $\beta = 0$ otherwise. This formula contains two parameters which reflect the effect produced by the characteristics of the polymer solution on the average-velocity profile. The value $u_{*\mathrm{cr}}$ characterizes the beginning of this effect, and $\beta \approx 0.174\,\alpha$ characterizes its intensity (the dimensionless parameter α is generally used in [2, 3]).

In a bounded region of variation of the variable ξ it is possible to use, instead of the logarithmic expression (1), a power expression which is close to it:

$$\langle u^+ \rangle \approx \gamma_n (z^+ u_*^\beta / u_{*\mathrm{cr}}^\beta)^{1/n}. \tag{2}$$

If, as is done in the description of the flow of a viscous liquid [1, 4], we use the distribution (2) in the entire pipe, then, integrating over a cross section, we arrive at a power law for the resistance:

$$\lambda \approx K \cdot \left(\frac{r u_{*\mathrm{cr}}}{\nu} \right)^{m_2} \mathrm{Re}^{-m_1},$$

$$m_1 = \frac{2 + 2\beta}{n + 1 + \beta}, \quad m_2 = \frac{2\beta}{n + 1 + \beta}. \tag{3}$$

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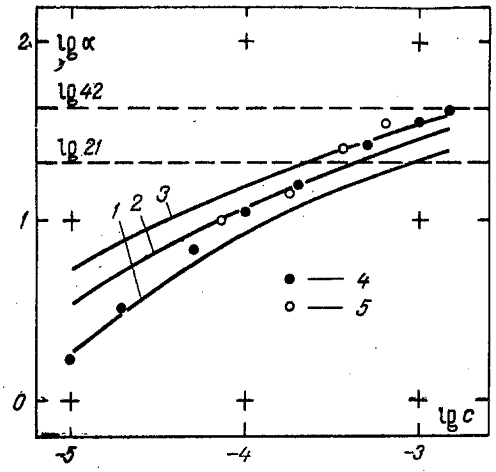

Fig. 1. Variation of the dimensionless parameter α as a function of c, the concentration of guar resin: 4) from the data of [5]; 5) from [6]; curves 1-3 were constructed by using power approximations of experimental data taken from [5] with n = 8, 10, and 12, respectively.

The explicit form of K = K(n, β) and the other similar coefficients encountered later will be given at the end of this article.

In the power law of resistance (3) the dependence on the diameter of the pipe, the viscosity of the solution, and the critical stress u^2_{*cr} for fixed Reynolds number enter only through the dimensionless complex ru_{*cr}/ν. At the same time, the coefficients m_1, m_2, K may depend† on other characteristics of the solution through β. Such a division of the effect in the case of a power approximation will in fact take place only in a bounded region of variation of the dimensionless parameters (not only the Reynolds number), as can be concluded from the conditions of the derivation of (3) and is illustrated below.

Equation (3) can be obtained directly by a power approximation of the "logarithmic" law of resistance, which follows [2, 3] from the distribution of the velocity (1)

$$\sqrt{8/\lambda} = 2{,}5\ln[0{,}256(\mathrm{Re}\sqrt{\lambda/32})^{1+\beta}(u_{*\mathrm{cr}}r/\nu)^{-\beta}] + 5{,}5. \quad (4)$$

Obviously, the suitable exponent n will depend on the value of the argument of the logarithm.

The expression (4) for a polymer solution is formally equivalent to the expression for the resistance of a viscous liquid with an effective Reynolds number $\mathrm{Re}(u_*/u_{*cr})^{\beta}$, which is greater than the actual value of Re for the flow in the pipe. However, to a larger Reynolds number there corresponds a larger value of n. To the expression (4) we can also assign the form of the law of resistance of a viscous liquid by means of a simple stretching of axes:

$$\lambda \to a\lambda,\ \mathrm{Re} \to b\,\mathrm{Re},$$
$$a = (1+\beta)^2,\quad b^{-1} = (1+\beta)\,[2{,}3u_{*\mathrm{cr}}r/\nu]^{\frac{\beta}{1+\beta}},$$

where the stretching coefficients do not depend on the position of the point in the λ, Re plane. Consequently, in the new coordinates the curves for the resistance of a viscous liquid and of polymer solutions will coincide, except for the region $\mathrm{Re} \approx \mathrm{Re}_{cr}$ and for very large Reynolds numbers, for which it is essential to consider the transitional characteristics and degradation of the polymer which are not described by the relation (4).

To a power function of the type (3), near the point at which the resistance begins to decrease, we can assign the form

$$\lambda/\lambda_{\mathrm{cr}} \approx (\mathrm{Re}_{\mathrm{cr}}/\mathrm{Re})^{m_1},\ \lambda_{\mathrm{cr}} \approx K_0\,\mathrm{Re}_{\mathrm{cr}}^{-\frac{2}{n+1}},$$
$$\mathrm{Re}_{\mathrm{cr}} \approx (\sqrt{32/K_0}\,ru_{*\mathrm{cr}}/\nu)^{\frac{n+1}{n}}. \quad (5)$$

Here the role of the parameters β, u_{*cr} is transferred to m_1, Re_{cr} [by using (5), we can easily give an inverse definition of β, u_{*cr} in terms of m_1, Re_{cr}].

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†In the subcritical regime of solution flow (when $u_* < u_{*cr}$) as for a viscous liquid, we will have $m_2 = 0$, $m_1 = 2/(n+1)$, $K = K_0 \equiv K(n, 0)$, and, in particular, when n = 7, we obtain the Blasius resistance law $\lambda \approx 0.32\,\mathrm{Re}^{-1/4}$.

In the special case $\mathrm{Re} \approx \mathrm{Re}_{\mathrm{cr}} \sim 10^4$, as is known, in the subcritical regime of flow a satisfactory description is obtained with n = 7. Then the equations become simpler:

$$\lambda \approx 0{,}32\,\mathrm{Re}_{\mathrm{cr}}^{1/4}\,(\mathrm{Re}/\mathrm{Re})^{\frac{2\beta+2}{\beta+8}},\ \mathrm{Re}_{\tilde{\mathrm{cr}}} \approx (10 r u_{*\mathrm{cr}}/\nu)^{\frac{8}{7}}. \tag{6}$$

For large Reynolds numbers we must select large values of n. However, it has already been noted that the choice of appropriate values of n is closely related to the value of the parameter β as well. This is clearly illustrated by the example given below.

In Fig. 1 we show the variation of the parameter $\alpha = 5.75\,\beta$ as a function of the concentration of a guar resin in water. The solid dots were plotted by using the experimental data of [5], which studied the flow of solutions over a wide range of concentrations of the guar resin JA-20-D. The open circles correspond to the data of [6]. The points in the figure represent values of α corresponding to a logarithmic approximation of the type (4) to the experimental curves of resistance for $\mathrm{Re} \sim 10^4$–10^5. It should be noted that the value of α may also be determined from the B_{max} tabulated in [5], using $\alpha \approx 1.63\,B_{\mathrm{max}}$.

The solid curves in Fig. 1 were obtained as follows. We drew straight lines through the experimental points on the graph of λ vs. Re taken from [5] for $\mathrm{Re} \sim 10^4$–10^5. Then we determined the exponents of the approximations to the true curves of resistance for various concentrations of the polymer. Using these exponents $m_1(c)$, we carried out the conversion (see (3)) of α(c) for the values n = 8, 10, and 12. Comparing the functions α(c) obtained in this manner and the previously plotted points in Fig. 1, we see that large values of n correspond better to large concentrations (large values of α).

It should be noted that the experimental points in Fig. 1 do not lie either on a curve of the type $\alpha \sim c$, discussed in [7], or on the curve $\alpha \sim \sqrt{c}$, discussed in [8-10].

Finally, according to [2, 9, 11, 12], we should expect the value of α to be bounded above. According to [2], the upper bound for α should be the value $\alpha_{\mathrm{max}}^{\theta} \approx 42$ if the dominant role in the phenomenon of resistance reduction is played by the time-dimension characteristic of the solution, or the value $\alpha^{l}_{\mathrm{max}} \approx 21$ in the case of the length-dimension characteristic. It can be seen from Fig. 1, in the first place, that for a guar resin concentration of $c = 1.5 \cdot 10^{-3}$ we reach the limiting value of $\alpha_{\mathrm{max}}^{\theta}$, and, in the second place, that for $c > 3 \cdot 10^{-4}$ we have the inequality $\alpha^{l}_{\mathrm{max}} < \alpha \leqslant \alpha^{\theta}_{\mathrm{max}}$. Thus, at least for a guar resin concentration by weight of more than $3 \cdot 10^{-4}$, the fundamental role in reducing the resistance must be played by the elastic properties of the liquid.

The same is true for other polymer solutions as well. The difference lies, as a rule, only in the different rate at which the limiting value of $\alpha_{\mathrm{max}}^{\theta}$ is reached (for polyoxyethylene solutions the limiting value is reached at concentrations which are lower by a factor of several tens than the concentrations of guar resin). In some solutions the limiting value is apparently lower than $\alpha_{\mathrm{max}}^{\theta}$ (see, for example, [5]).

Since the quantity β is bounded above and since n > 7 (with large values of n corresponding to larger values of β), as follows from (3), the curves of lowered resistance $\lambda = \lambda(\mathrm{Re}, \beta, \ldots)$ cannot have at any point a slope steeper than $\sim \mathrm{Re}^{-1}$. Moreover, as the Reynolds number increases, the slope of such curves becomes less and less steep. Such an interpretation of the limiting regime of resistance reduction was mentioned in [13].

Limiting Asymptote of Resistance Reduction. Many experimental data seem to support the assumption that there is saturation of the effect of polymers on turbulent flows. According to [2], under conditions of such saturation the velocity profile far from the walls and the axis of the pipe has the form

$$\langle u^+ \rangle = 2.5 \ln \eta + 5.5,\ \eta = 1.35 \cdot 10^{-4} (z^+)^{4{,}65}. \tag{7}$$

Making use, as before, of a power approximation to this expression

$$\langle u^+ \rangle \approx 10^{-4/n}\, \gamma_n^{\,r} (z^+)^{4{,}65/n}, \tag{8}$$

we arrive at a power law for the resistance

$$\lambda \approx K_0 10^{\frac{8}{n+1}} \mathrm{Re}^{-m_\infty}, \; m_\infty = 2(1 + n/4{,}65)^{-1}. \tag{9}$$

Since the limiting profile (7) is independent of the specific properties of the polymer solution, the law of minimum resistance is found to be a universal law, i.e., its form does not depend on the characteristics of the polymer or the diameter of the pipe.

It can be seen from Fig. 1 that a value of $n \approx 12$ corresponds well to the curve showing the effectiveness of the polymer as a function of concentration for large values α. If we now substitute this value into Eq. (9), we obtain

$$\lambda \approx C \, \mathrm{Re}^{-0{,}56}, \; C \sim 1. \tag{10}$$

It is interesting to compare this relation with the expressions found empirically. The following power approximations have been proposed for the curve of minimal resistance: $\lambda \sim \mathrm{Re}^{-1/2}$ in [14], $\lambda \approx 1.68 \, \mathrm{Re}^{-0.55}$ in [15], $\lambda \approx 2.18 \, \mathrm{Re}^{-0.57}$ for $\mathrm{Re} < 4 \cdot 10^4$ and $\lambda \approx 0.27 \, \mathrm{Re}^{-0.37}$ for $\mathrm{Re} > 4 \cdot 10^4$ in [16], $\lambda \approx 5.6 \, \mathrm{Re}^{-2/3}$ in [17], and, lastly, $\lambda \approx 0.5 \, \mathrm{Re}^{-0.43}$ in [18].

The minimal resistance is characterized by the fact that, like the resistance of a viscous liquid, it is determined by the single parameter Re. Therefore we can speak more definitely of the value of the exponent n corresponding to some particular region of Reynolds numbers. According to [2], the equation of the minimum-resistance curve has the form

$$\sqrt{8/\lambda} \approx 11{,}6 \ln(\mathrm{Re}\sqrt{\lambda/32}) - 32. \tag{11}$$

The function appearing on the right side, together with its derivative, is well approximated for $\mathrm{Re} \sim 10^4$ by the power function

$$\lambda \approx 2{,}4 \, \mathrm{Re}^{-0{,}585}. \tag{12}$$

For $\mathrm{Re} \sim 10^5$ we have

$$\lambda \approx 0{,}3 \, \mathrm{Re}^{-0{,}38}. \tag{13}$$

Turbulent Boundary Layer with No Pressure Gradient. In the problem of longitudinal flow past a flat plate, we shall assume, as is usually done in a description of the flow of a viscous liquid [1, 4], that the turbulent boundary layer begins at the forward edge of the plate, and that we can neglect the details of the velocity distribution in the viscous substratum and the transition zone near the plate; we use the approximate integral relation

$$(u_*/V)^2 = \partial\delta_2/\partial x, \quad \delta_2 = V^{-2} \int_0^\delta \langle u \rangle (V - \langle u \rangle) \, dz. \tag{14}$$

Here, unlike flow in a pipe, the magnitudes of the velocity and length vary along the plate as the boundary layer develops. As usual [1, 4], we shall assume that the distributions at each cross section are determined by the local values of $u_*(x)$, $\delta(x)$, and we shall use the same formulas as in the case of a pipe, replacing r with δ.

For the power approximation of the distribution of the average velocity of the flow of a polymer solution in the entire boundary layer, we obtain from Eq. (2) the relations

$$\langle u \rangle / V = (z/\delta)^{1/n}, \quad \delta_2/\delta = \frac{n}{(n+1)(n+2)}, \tag{15}$$

which do not differ from such relations for the case of a viscous liquid.

From (2) we also obtain the relation between $\delta(x)$ and $u_*(x)$:

$$\delta = \nu u_{*\mathrm{cr}}^{\beta} (V/\gamma_n)^n u_*^{-(n+1+\beta)}. \tag{16}$$

Using relations (15), (16), we can transform Eq. (14) into an equation for $u_*(x)$:

$$\frac{du_*(x)}{dx} = -K_1 (\nu u_{*\mathrm{cr}}^{\beta} V^{n+2})^{-1} [u_*(x)]^{n+4+\beta}. \tag{17}$$

According to this equation, the stress on the plate, u_*^2, decreases monotonically as we move away from the forward edge of the plate. If the plate is sufficiently long, at some distance from the forward edge the stress may become lower than the critical stress for the beginning of the polymer effect, $u_{*\mathrm{cr}}^2$, i.e., starting from some $x = L_{\mathrm{cr}}$, the polymer begins to have an effect on the development of the boundary layer. Let us consider separately two possible cases, with $L > L_{\mathrm{cr}}$ and $L < L_{\mathrm{cr}}$.

"Small" Plate (Large Velocities). When the length of the plate is less than the critical length ($L < L_{\mathrm{cr}}$), the polymer changes the character of the boundary layer at all distances from the edge ($\beta > 0$ everywhere). Integrating the equation with $u_* \to \infty$ as $x \to 0$, we obtain the equation

$$u_*(x) = u_{*\mathrm{cr}} (L_{\mathrm{cr}}/x)^{p_1}, \quad p_1 = (n+3+\beta)^{-1}, \tag{18}$$

in which

$$L_{\mathrm{cr}} = K_2 \frac{\nu}{u_{*\mathrm{cr}}} \left(\frac{V}{u_{*\mathrm{cr}}} \right)^{n+2}. \tag{19}$$

We also have, according to (16),

$$\delta(x) = \frac{\nu}{u_{*\mathrm{cr}}} \left(\frac{V}{\gamma_n u_{*\mathrm{cr}}} \right)^n \left(\frac{x}{L_{\mathrm{cr}}} \right)^{p_2}, \quad p_2 = \frac{n+1+\beta}{n+3+\beta}. \tag{20}$$

From Eqs. (18), (20) it can be seen that in polymer solutions the stress on the walls decreases somewhat more slowly and the boundary layer grows more rapidly than in the case of a viscous liquid.

Introducing the local coefficient of resistance $c_f = 2(u_*/V)^2$ and the Reynolds number $\mathrm{Re}_x = Vx/\nu$, we can rewrite Eq. (18) in the form

$$c_f = K_3 (u_{*\mathrm{cr}}/V)^{2\beta p_1} \mathrm{Re}_x^{-2p_1}. \tag{21}$$

The equations for the complete coefficient of resistance $C_f = L^{-1}\int_0^L c_f dx$ can be obtained by another integration.

$$C_f = K_4 \left(\frac{u_{*\mathrm{cr}}}{V} \right)^{2\beta p_1} \mathrm{Re}_L^{-2p_1}; \tag{22}$$

$$C_f = K_4 \left(\frac{u_{*\mathrm{cr}} L}{\nu} \right)^{2\beta p_1} \mathrm{Re}_L^{-2(1+\beta)p_1}. \tag{23}$$

The same relation is written in two different ways because it is possible to specify different conditions. The first equation is suitable for conditions in which we have a specified velocity of oncoming flow V, while the second is suitable for a fixed plate length L and a variable velocity V. The appearance of two different written forms of the resistance law for a polymer solution which are reducible to one form for a viscous liquid ($\beta = 0$) is

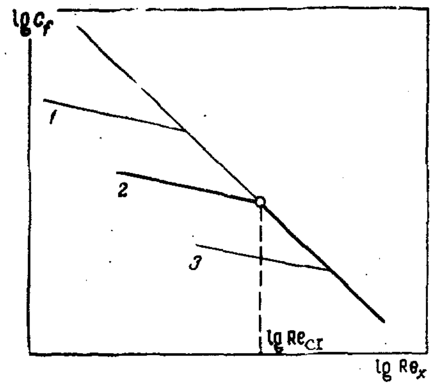

Fig. 2. Curves of local resistance of the plate (1, 2, 3) for different velocities of the oncoming flow $V_1 < V_2 < V_3$ (turbulent regime of flow past the plate): 1) V_1; 2) V_2; 3) V_3.

due to the dimensional characteristic of the solution $u_{*\mathrm{cr}}$. Because of this, we have the additional dimensionless characteristics $u_{*\mathrm{cr}}/V$, $u_{*\mathrm{cr}}L/\nu$.

The condition for a small plate, $L < L_{\mathrm{cr}}$, can be rewritten, with the aid of (19), as

$$L < K_2^{\frac{1}{n+3}} \nu/u_{*\mathrm{cr}} \mathrm{Re}_L^{\frac{n+2}{n+3}}, \tag{24}$$

$$V > K_2^{\frac{-1}{n+3}} u_{*\mathrm{cr}} \mathrm{Re}_L^{\frac{1}{n+3}}. \tag{25}$$

The last inequality is satisfied in practice for $V > 50\, u_{*\mathrm{cr}}$. Usually $u_{*\mathrm{cr}} < 10^{-1}$ m/sec, and consequently, when $V > 5$ m/sec, the plate may be considered short in the above-described sense of the word.

Large Plate (Low Velocities). According to (19), when the flow velocity is reduced, the value of L_{cr} decreases extremely fast and may become so small that we will have the inequality $L_{\mathrm{cr}} < L$. In this case of a "long" plate, integration of Eq. (17) when $x < L_{\mathrm{cr}}$ leads to Eq. (18), whereas when $x > L_{\mathrm{cr}}$, we obtain

$$x - x_{\mathrm{cr}} = K_{20} \frac{\nu}{V} \left(\frac{V}{u_*}\right)^{n+3}, \quad x_{\mathrm{cr}} = \frac{2\beta}{(n+3)(n+1+\beta)} L_{\mathrm{cr}}. \tag{26}$$

This relation is completely analogous to the equation for $u_*(x)$ in a viscous liquid, except that in the present case the origin from which x is measured is displaced† to the point x_{cr}. For the values we are considering, $n > 7$, $\beta < 7.3$, the inequality $x_{\mathrm{cr}} < 0.1\, L_{\mathrm{cr}}$ holds, i.e., the value of x_{cr} is always small in comparison with x ($x > L_{\mathrm{cr}}$).

Neglecting the value of x_{cr}, we can rewrite (26) as a relation which coincides in form with the law of resistance of a viscous liquid:

$$c_f \approx K_{30} \mathrm{Re}_x^{-\frac{2}{n+3}}, \quad \mathrm{Re}_x > \mathrm{Re}_{\mathrm{cr}} \tag{27}$$

Thus, at large distances from the forward edge of the plate ($x > L_{\mathrm{cr}}$) the development of the boundary layer on a long plate takes place practically in the same way as in a viscous liquid.

An approximate estimate for the critical value of the Reynolds number, $\mathrm{Re}_{\mathrm{cr}}$, at which the flow past the plate goes from one regime to the other has been found by setting the expressions (21) and (27) equal to each other:

$$\mathrm{Re}_{\mathrm{cr}} \approx K_5 (V/u_{*\mathrm{cr}})^{n+3}, \tag{28}$$

which differs by a numerical factor of the order of unity from the exact value

$$\mathrm{Re}_{\mathrm{cr}} = K_2 (V/u_{*\mathrm{cr}})^{n+3}. \tag{29}$$

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†At the point $x = x_{\mathrm{cr}}$, itself the boundary layer has no singularity, since Eq. (26) described the development of the boundary layer only for $x > L_{\mathrm{cr}} > x_{\mathrm{cr}}$.

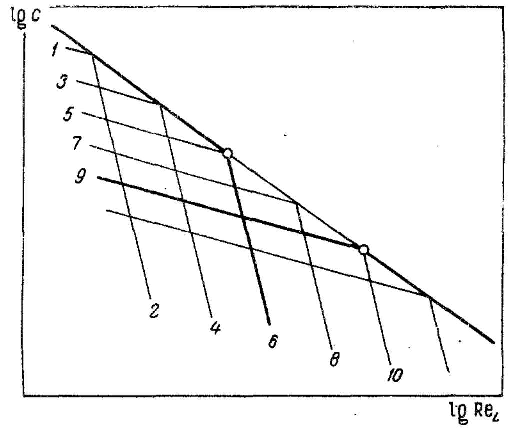

Fig. 3. Curves of overall resistance for various specified values of the velocity of the oncoming flow (1, 3, 5, ...) and various lengths of the plate (2, 4, 6, ...): $V_1 < V_3 < V_5$ ..., $L_3 < L_4 < L_6 <$ .... The ordinate axis shows for C_f.

This discrepancy is the result of neglecting x_{cr}.

The character of the transition from the "viscous" regime of flow past the plate to the regime of flow with reduced resistance is different from the character of the transition usually encountered in the case of the flow of solutions in pipes. The resistance decreases for small Reynolds numbers Re_x [see (21)], whereas for "viscous" flow the resistance decreases for large values of the Reynolds number [see (27)]. This is illustrated graphically in Fig. 2.

By integrating (21) and (27) along the length of the plate, we can easily convince ourselves that even for an overall coefficient of resistance, with $\mathrm{Re}_L > \mathrm{Re}_{cr}$, we have the law of resistance of a viscous liquid

$$C_f \approx K_{40}\,\mathrm{Re}_L^{-\frac{2}{n+3}},\quad \mathrm{Re}_L > \mathrm{Re}_{cr}. \tag{30}$$

The curves of overall resistance are shown in Fig. 3. They consist of rectilinear segments (in the power approximation considered here with constant n) of viscous and reduced resistance, where the order of succession of these segments is different, depending on whether flow velocities or plate lengths are specified. On the curves with fixed length L the "viscous" segment precedes the segment with reduced resistance, whereas the contrary is true for a fixed velocity V. The entire region of reduced resistance can be plotted with two intersecting families of parallel straight lines, with V = const and L = const (actually these straight lines are close to the true curves only in a small region of Reynolds number values). The curves of resistance of a plate of specified length have been discussed earlier in [19], and the curves of the other type have been discussed in [20, 21].

In [19, 20] the resistance formulas were derived by using an expansion which is not valid when the resistance has been greatly reduced. In [20] an error was made in the integration of the equation for $u_*(x)$. A somewhat different analysis of power approximations in the case of a short plate is given in [22].

In conclusion, we give below the expressions for the coefficients $K_v = K_v(n, \beta)$ encountered above in terms of the quantities n, γ_n, and β:

$$\begin{gathered}
K = K_0\,(32/K_0)^{m_2/2},\quad K_0 = 2^{\frac{5+n}{1+n}}\left[\frac{(n+1)(2n+1)}{n^2\gamma_n}\right]^{\frac{2n}{n+1}},\\
K_1 = \frac{(n+1)(n+2)}{n\,(n+1+\beta)}\,\gamma_n^n,\quad K_2^{-1} = (n+3+\beta)\,K_1,\\
K_3 = 2K_2^{2p_1},\quad K_4 = \frac{n+3+\beta}{n+1+\beta}\,K_3,\\
K_5^{2\beta p_1} = (K_{30}/K_3)^{n+3},\quad K_{v0} \equiv K_v\,(n,\ 0).
\end{gathered} \tag{31}$$

For $n \approx 10$ and $\beta \approx 5$ we have the following estimates for these coefficients: $\gamma_{10} \approx 11.5$; $K_{10} \approx 1.5K_1 \approx 5 \cdot 10^{10}$; $K_2 \approx K_{20} \approx 10^{-12}$; $K_5 \sim 10^{-12}$; $K_3 \approx K_4 \approx 3K_{30} \approx 3K_{40} \approx 10^{-1} \approx K_0$; $K \approx 1$.

A simple estimate for the exponents in the power law of resistance for a pipe (3) can be obtained if in the power approximation of the right side of (4) we assume that the functions and their first derivatives are close to each other:

$$m_1 = \frac{1+\beta}{\beta}\,m_2 \approx 2\,\frac{0{,}88\,(1+\beta)\sqrt{\lambda}}{1+0{,}88\,(1+\beta)\sqrt{\lambda}}. \tag{32}$$

For a viscous liquid, as is known, this equation holds satisfactorily with a numerical coefficient of 1.0 instead of 0.88.

The limiting value $\beta \approx 7.3$, according to this equation there corresponds

$$m_1 \approx \frac{14.6\sqrt{\lambda}}{1+7.3\sqrt{\lambda}}, \tag{33}$$

but the indicated upper bound m_1 is valid only for resistance curves which lie above the curve of minimal resistance (the limiting asymptote).

For the minimal resistance curve (with a corresponding β value of $\beta \approx 3.65$) we obtain

$$m_\infty = \frac{8.2\sqrt{\lambda}}{1+4.1\sqrt{\lambda}}, \tag{34}$$

i.e., an even stronger bound.

NOTATION

$\langle u \rangle$, average velocity; z, distance for wall; r, radius of pipe; δ, thickness of boundary layer; λ, coefficient of resistance; Re, Reynolds number; $\alpha = 5.75\beta$, dimensionless characteristic of polymer solution; $u^2{}_{*cr}$, critical stress at which the polymer begins to affect the turbulence (density of solution is taken to be unity); ν, viscosity of solution; u_*, dynamic velocity; $\langle u^+ \rangle = \langle u \rangle / u_*$, $z^+ = z u_* / \nu$; V, maximum velocity; L, length of plate; c_f, C_f, local and overall coefficients of resistances; n, γ_n, K_ν, numerical coefficients.

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